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ON THE MINIMAL SPEED DETERMINACY OF TRAVELING WAVES TO A THREE-SPECIES COMPETITION MODEL*

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Abstract. The minimal wave speed determinacy of traveling wave solution to a competitive three-species system is investigated. Particularly, we are concerned about the speed at which a strong competitor species invades other two residents via the diffusive Lotka-Volterra model. By considering the non-dimensional cooperative version of the model, we apply the upper-lower solution method to study the traveling wave speed. New upper solutions are established to derive sufficient conditions on the problem parameters so that the minimal wave speed is given by the speed of the corresponding linearized system. Our new results are compared with those in the previous studies and some improvements are provided.

Keywords: Three species competition, biological invasion, traveling waves, speed determinacy.

AMS Subject Classification: 35K57, 35C07, 92D25.

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1 Introduction

In an ecosystem, species that share the living space compete with each other seeking existence and growth. Naturally, this depends on their competition characteristics and the validity of environmental resources (Kolar & Lodge, 2001; Shea & Chesson, 2002; Sher & Hyatt, 1999). These competitive species may coexist or some of them can not survive and die out (Amarasekare, 2003; Yu & Wilson, 2001). Competition of two species in the same area was extensively studied (Alhasanat & Ou, 2019a,b, 2020a,b; Holzer & Scheel, 2012; Huang, 2010; Li et al., 2005; Lewis et al., 2002; Ma et al., 2020; Roques et al., 2015; Yue et al., 2020), where significant results and remarks were provided. Studies become complicated and results can not be easily generalized for more than two interacted species. We consider here the interaction of three species so that a new competitor is introduced into the living space of two species. We assume that the resident species have different environmental resource preferences, but the new one strongly competes with both of them, which indicates a biological invasion. In particular, we are interested in the rate at which the invader species spreads into the resident species' environment. This leads to a full understanding of the invasion behavior and control it. Possible coexistence for such interactions was investigated in literature based on the competition characteristics (Chen et al., 2013; Kan-on & Mimura, 1998; Mimura & Tohma, 2015).

Mathematically, for time τ and location y, let $\psi_1(\tau, y)$ be the density of an invader of two resident species with densities $\psi_2(\tau, y)$ and $\psi_3(\tau, y)$. The species growth follows the following

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diffusive Lotka-Volterra model:

$$\begin{cases} \psi_{1,\tau} = D_1\psi_{1,yy} + r_1\psi_1\left(1 - \dfrac{\psi_1}{N_1} - B_{12}\psi_2 - B_{13}\psi_3\right), \\ \psi_{2,\tau} = D_2\psi_{2,yy} + r_2\psi_2\left(1 - B_{21}\psi_1 - \dfrac{\psi_2}{N_2}\right), \\ \psi_{3,\tau} = D_3\psi_{3,yy} + r_3\psi_3\left(1 - B_{31}\psi_1 - \dfrac{\psi_3}{N_3}\right), \end{cases} \tag{1}$$

where N_i is the carrying capacity which includes the intraspecific competition, D_i is the diffusion coefficient, and r_i is the growth rate of species i, for $i = 1,2,3$; B_{1j} and B_{j1}, for $j = 2,3$, are the interspecific competition coefficients of the first species with the other two species. The non-dimensional cooperative version of the model (1) is derived by using the transformation

$$t = r_1\tau,\ x = y\sqrt{r_1/D_1},\quad u = \psi_1/N_1,\ v = 1 - \psi_2/N_2,\ w = 1 - \psi_3/N_3.$$

By defining, for $i = 1,2,3$ and $j = 2,3$

$$d_i = D_i/D_1,\ \alpha_i = r_i/r_1,\ B_{1j}N_j = b_{1j},\ B_{j1}N_1 = b_{j1},$$

the system of the new functions $(u,v,w)(t,x)$ reads

$$\begin{cases} u_t = u_{xx} + u(1 - b_{12} - b_{13} - u + b_{12}v + b_{13}w), \\ v_t = d_2v_{xx} + \alpha_2(1-v)(b_{21}u - v), \\ w_t = d_3w_{xx} + \alpha_3(1-w)(b_{31}u - w). \end{cases} \tag{2}$$

Our analysis will be carried out on the last model, where any result can be easily rewritten in terms of the original model. It is clear that domain of (u,v,w) is inside the domain

$$\chi = \{(\vartheta_1,\vartheta_2,\vartheta_3)|(0,0,0) \le (\vartheta_1,\vartheta_2,\vartheta_3) \le (1,1,1)\}.$$

As mentioned above, we assume that the invader species is a strong competitor compared with the residents, that is, we let

$$b_{12} + b_{13} < 1,\quad b_{21} > 1,\quad b_{31} > 1. \tag{3}$$

By computing the possible fixed point solutions of the kinetic system associated with the system (2) inside χ, taking into account the condition (3), we get

$$(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1),\ (1,1,1).$$

Furthermore, by standard linearization, $(u,v,w) = (0,0,0)$ is unstable state and $(u,v,w) = (1,1,1)$ is stable. Originally, the former state means that there are no individuals of ψ_1 but both of ψ_2 and ψ_3 are at their maximum values N_2 and N_3, while the latter state means that ψ_1 wins the competition and both of ψ_2 and ψ_3 die out. The traveling wave solution in the form

$$(u,v,w)(t,x) = (U,V,W)(z),\quad z = x - ct \tag{4}$$

that connects $(1,1,1)$ and $(0,0,0)$, is used to study this invasion phenomenon and determine the species spreading speed. Here, z is called the wave variable, $c \ge 0$ is the wave speed, and (U,V,W) is called the wave profile. For existence of traveling waves and their properties for three species competition models, we refer to the work of Chen et al. (2013, 2012); Guo et al. (2015); Hou & Li (2017); Pan et al. (2021).

The traveling wave system of the wave profile is found by substituting (4) into the system (2), which leads to get

$$\begin{cases} L_u[U,V,W] & := U'' + cU' + U(1 - b_{12} - b_{13} - U + b_{12}V + b_{13}W) = 0, \\ L_v[U,V] & := d_2V'' + cV' + \alpha_2(1 - V)(b_{21}U - V) = 0, \\ L_w[U,W] & := d_3W'' + cW' + \alpha_3(1 - W)(b_{31}U - W) = 0, \end{cases} \tag{5}$$

subject to

$$(U,V,W)(-\infty) = (1,1,1) \quad \text{and} \quad (U,V,W)(+\infty) = (0,0,0). \tag{6}$$

By Guo et al. (2015); Pan et al. (2021), it was proved that there exists $c_{\min} \geq 0$ (called the minimal wave speed) so that a monotone decreasing traveling wave solution to (5)-(6) exists if and only if $c \geq c_{\min}$. Furthermore, it was shown that this minimal speed is the asymptotic spreading speed, the speed at which the invader species $u(t,x)$ eventually approaches the state where it wins the competition. The formula of $c_{\min}$ is not valid in general. However, we should have

$$c_{\min} \geq c_0 = \sqrt{1 - b_{12} - b_{13}}$$

so that the wave profile is monotone. The formula of c_0 is obtained by linearizing the system (5) at the null fixed point. See the work of Pan et al. (2021) for more details. When $c_{\min} = c_0$, we say that the minimal wave speed of the system (5) is linearly determined which leads to giving an explicit formula of it. On the other hand, when $c_{\min} > c_0$, we say that it is nonlinearly determined.

Guo et al. (2015) proved that the minimal wave speed is linearly determined when $0 < d_j < 0$, for $j = 2,3$, and one of the following occurs:

$$b_{j1}(b_{12} + b_{13}) \leq 1 \tag{7}$$

or

$$\begin{cases} b_{j1}(b_{12} + b_{13}) > 1, \\ \alpha_j < \dfrac{(2 - d_j)(1 - b_{12} - b_{13})}{b_{j1}(b_{12} + b_{13}) - 1}. \end{cases} \tag{8}$$

Pan et al. (2021) studied the minimal speed determinacy by using the upper-lower solution method, and showed that it is linearly determined when $0 < d_j < 0$, for $j = 2,3$, and one of the following occurs:

$$-2(1 - b_{12} - b_{13}) + b_{12}b_{21} + b_{13}b_{31} \leq 0 \tag{9}$$

or

$$\begin{cases} b_{12} < \dfrac{1 - b_{13}}{2}, \; b_{13} < \dfrac{1 - b_{12}}{2}, \\ \alpha_j < \dfrac{(2 - d_j)(1 - b_{12} - b_{13})^2}{b_{1j}b_{j1} - (1 - b_{12} - b_{13})}. \end{cases} \tag{10}$$

Moreover, Pan et al. (2021) provided a result for a larger domain of d_2 and d_3 so that the minimal speed is linearly determined when

$$\begin{cases} 0 < d_j < 2 + \dfrac{\alpha_j}{1 - b_{12} - b_{13}}, \\ \dfrac{\alpha_j - (d_j - 2)(1 - b_{12} - b_{13})}{\alpha_j b_{j1}} > \max\left\{ \dfrac{d_j - 2}{2d_j}, \dfrac{b_{12} + b_{13}}{2(1 - b_{12} - b_{13})} \right\}, \quad \text{for } j = 2,3. \end{cases} \tag{11}$$

The purpose of this work is to revisit the minimal wave speed determinacy by using the upper-lower solution method. We successfully derive new sufficient conditions for the linear speed determinacy by constructing suitable upper solution to the system (5). We compare our results with that in (7)-(11), and provide some improvements.

The rest of the paper is organized as follows. In Section 2, we present the solution behavior near the null fixed point which plays a main role in constructing the required upper solutions. Also, we introduce the upper-lower solution method. The main results are given in Section 3, and conclusions are presented in Section 4.

2 Preliminaries

Let's first introduce the derivation of the traveling wave solution behavior near the null fixed point. Details can be found in the work of Pan et al. (2021). When $c \geq c_{\min}$, substitute $(U, V, W)(z) = (\zeta_1, \zeta_2, \zeta_3)e^{-\mu z}$, for positive constants $\zeta_1, \zeta_2, \zeta_3$, and μ, into the traveling wave system (5) and linearize the resulted system to get an algebraic system in terms of μ. This system has a non-trivial solution if μ equals one of the following positive values

$$\begin{aligned}
\mu_1 &= \frac{1}{2}\left(c - \sqrt{c^2 - 4(1 - b_{12} - b_{13})}\right), \quad & \mu_2 &= \frac{1}{2}\left(c + \sqrt{c^2 - 4(1 - b_{12} - b_{13})}\right) \\
\mu_3 &= \frac{1}{2d_2}\left(c + \sqrt{c^2 + 4d_2\alpha_2}\right), & \mu_4 &= \frac{1}{2d_3}\left(c + \sqrt{c^2 + 4d_3\alpha_3}\right).
\end{aligned} \tag{12}$$

By substituting $\mu_k,\ k = 1, 2, 3, 4$ into the linearized system, we can find values of $\zeta_m,\ m = 1, 2, 3$ associated with each μ_k. Indeed the solution behavior near infinity is given by

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix}(z) = C_1 \begin{pmatrix} 1 \\ \zeta_{12} \\ \zeta_{13} \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} 1 \\ \zeta_{22} \\ \zeta_{23} \end{pmatrix} e^{-\mu_2 z} + C_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-\mu_3 z} + C_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-\mu_4 z}$$

for constants $C_1, C_3, C_4 > 0$ or $C_1 = 0$ with $C_2, C_3, C_4 > 0$, where

$$\begin{aligned}
\zeta_{12} &= \frac{-\alpha_2 b_{21}}{d_2\mu_1^2 - c\mu_1 - \alpha_2}, \quad & \zeta_{13} &= \frac{-\alpha_3 b_{31}}{d_3\mu_1^2 - c\mu_1 - \alpha_3}, \\
\zeta_{22} &= \frac{-\alpha_2 b_{21}}{d_2\mu_2^2 - c\mu_2 - \alpha_2}, & \zeta_{23} &= \frac{-\alpha_3 b_{31}}{d_3\mu_2^2 - c\mu_2 - \alpha_3}.
\end{aligned}$$

Now, we give the definition of the upper solution to the system (5) following that was given by Alhasanat & Ou (2020a) for the two-species model.

Definition 1. *If $(\overline{U}, \overline{V}, \overline{W})(z)$ is continuous on $(-\infty, \infty)$ and differentiable except at n finite number of points $z_q, q = 1, 2, 3, .., n$, satisfying $(\overline{U}', \overline{V}', \overline{W}')(z_q^-) \geq (\overline{U}', \overline{V}', \overline{W}')(z_q^+)$ and $L_u[\overline{U}, \overline{V}, \overline{W}](z) \leq 0,\ L_v[\overline{U}, \overline{V}](z) \leq 0,\ L_w[\overline{U}, \overline{W}](z) \leq 0,$ for $z \neq z_q$, then $(\overline{U}, \overline{V}, \overline{W})(z)$ is an upper solution to the problem (5).*

The lower solution is defined similarly by reversing the inequalities in the above definition of the upper solution. However, the conditions for the linear speed selection will be derived by constructing only suitable upper solutions based on the following lemma (Pan et al., 2021, Theorem 3.3).

Lemma 1. *If there exists a monotone upper solution $(\overline{U}, \overline{V}, \overline{W})(z)$, with $c = c_0$, to the problem (5) so that $0 < \overline{U}(-\infty) \leq 1$ and $\overline{U}(+\infty) = 0$, then the minimal wave speed is linearly determined.*

3 The linear determination of the minimal wave speed

When the problem parameters satisfy specific conditions, we derive the following results for the linear determination of the minimal wave speed, $c_{\min}$, of (5). This is done by constructing suitable upper solutions that share the same behavior of the solution near the null fixed point and applying Lemma 1. Domain of the diffusion coefficients will be considered as

$$0 < d_j \leq 2, \text{ for } j = 2, 3. \tag{13}$$

We start first by driving a result that is independent of the growth rates α_2 and α_3 and then we compare it with similar previous results.

Theorem 1. *When (3) and (13) hold, the minimal speed of the system (5) is linearly determined if the following conditions satisfy:*

$$2(2-b_{12}-b_{13})(b_{j1}-1) \leq (1-b_{12}-b_{13})\left(1+4\sqrt{b_{j1}(b_{j1}-1)}\right), \ j=2,3, \tag{14}$$

$$b_{12}(b_{21}-1)+b_{13}(b_{31}-1) < (1-b_{12}-b_{13})\left(\frac{1}{2}+\sqrt{1-1/b_{\max}}\right), \tag{15}$$

where $b_{\max} = \max\{b_{21}, b_{31}\}$.

Proof. To construct the desired upper solution to the system (5) when $c = c_0$, define $\overline{U}(z) \in [0,1]$ as the monotone solution of

$$\overline{U}'(z) = -\mu_1\overline{U}(z)(1-\overline{U}(z))^{\frac{1}{2}}, \ \ \overline{U}(-\infty)=1, \ \ \overline{U}(+\infty)=0, \tag{16}$$

and let

$$\overline{V}(z) = \min\{1, b_{21}\overline{U}(z)\} = \begin{cases} 1, & z \leq z_1, \\ b_{21}\overline{U}(z), & z > z_1, \end{cases}$$

$$\overline{W}(z) = \min\{1, b_{31}\overline{U}(z)\} = \begin{cases} 1, & z \leq z_2, \\ b_{31}\overline{U}(z), & z > z_2, \end{cases}$$

where $1 = b_{21}\overline{U}(z_1)$, $1 = b_{31}\overline{U}(z_2)$, and μ_1 is defined in (12) with $\mu_1(c_0) = c_0/2 = \sqrt{1-b_{12}-b_{13}}$. Notice that $\overline{V}(z)$ and $\overline{W}(z)$ are continuous and differentiable except at z_1 and z_2, receptively, with

$$\overline{V}'(z_1^-) = 0 \geq b_{21}\overline{U}'(z_1^+) = \overline{V}'(z_1^+)$$

and

$$\overline{W}'(z_2^-) = 0 \geq b_{31}\overline{U}'(z_2^+) = \overline{W}'(z_2^+).$$

Hence, $(\overline{U}, \overline{V}, \overline{W})(z)$ satisfies the first inequality of Definition 1 for the non-differentiation points. For the other inequalities, when $z \neq z_{1,2}$, we consider the case when $b_{21} \leq b_{31}$. This, with monotonicity of $\overline{U}$, implies that $z_1 \leq z_2$. The proof of the other case is the same and omitted.

First of all, we compute

$$\overline{U}'' = \mu_1^2\overline{U}(1-\overline{U}) - \frac{1}{2}\mu_1^2\overline{U}^2.$$

Now, the argument will split into the following three cases:
Case 1: When $z < z_1$. The facts $L_v[\overline{U}, \overline{V}](z) = L_v[\overline{U}, 1](z) = 0$ and $L_w[\overline{U}, \overline{W}](z) = L_w[\overline{U}, 1](z) = 0$ are trivially satisfied. For the U-equation, we have

$$\begin{aligned} L_u[\overline{U}, \overline{V}, \overline{W}] &= \overline{U}'' + c\overline{U}' + \overline{U}(1-\overline{U}) \\ &= \overline{U}(1-\overline{U})\left(\mu_1^2 - \frac{\mu_1^2\overline{U}}{2(1-\overline{U})} - \frac{c\mu_1}{(1-\overline{U})^{\frac{1}{2}}} + 1\right) \\ &:= \overline{U}(1-\overline{U})f(\overline{U}). \end{aligned}$$

By substituting the values of c_0 and $\mu_1(c_0)$, $f(\overline{U})$ is re-defined as

$$f(\overline{U}) = 2 - b_{12} - b_{13} - \frac{(1-b_{12}-b_{13})\overline{U}}{2(1-\overline{U})} - \frac{2(1-b_{12}-b_{13})}{(1-\overline{U})^{\frac{1}{2}}},$$

for $\overline{U} \in (1/b_{21}, 1]$ as $z < z_1$. Trivial computations yield

$$f'(\overline{U}) = \frac{-(1-b_{12}-b_{13})(1+2(1-\overline{U})^{\frac{1}{2}})}{2(1-\overline{U})^2} < 0.$$

Hence,

$$\begin{aligned}\sup_{\overline{U}\in(1/b_{21},1]} f(\overline{U}) &= f(1/b_{21})\\ &= \frac{2(b_{21}-1)(2-b_{12}-b_{13})-(1-b_{12}-b_{13})(1+4\sqrt{b_{21}(b_{21}-1)})}{2(b_{21}-1)}\\ &\le 0,\end{aligned}$$

by condition (14), that is, $L_u[\overline{U},\overline{V},\overline{W}](z)\le 0$ for all $z<z_1$.

Case 2: When $z_1<z<z_2$. We still have $L_w[\overline{U},1](z)=0$. For the V-equation, we compute

$$L_v[\overline{U},\overline{V}] = d_2\overline{V}''+c\overline{V}' = d_2b_{21}\mu_1^2\overline{U}(1-\overline{U})-\frac{d_2b_{21}}{2}\mu_1^2\overline{U}^2-cb_{21}\mu_1\overline{U}(1-\overline{U})^{\frac{1}{2}}.$$

By substituting $c=c_0$ and using $-(1-\overline{U})^{\frac{1}{2}}\le-(1-\overline{U})$, we get

$$L_v[\overline{U},\overline{V}]\le b_{21}(1-b_{12}-b_{13})(d_2-2)\overline{U}(1-\overline{U})-\frac{d_2b_{21}}{2}\mu_1^2\overline{U}^2.$$

From (13), we have $L_v[\overline{U},\overline{V}](z)\le 0$, for $z_1<z<z_2$. Now,

$$\begin{aligned}L_u[\overline{U},\overline{V},\overline{W}] &= \overline{U}''+c\overline{U}'+\overline{U}(1-b_{12}-\overline{U}+b_{12}b_{21}\overline{U})\\ &= \overline{U}(1-\overline{U})\left(\mu_1^2-\frac{\mu_1^2\overline{U}}{2(1-\overline{U})}-\frac{c\mu_1}{(1-\overline{U})^{\frac{1}{2}}}-\frac{b_{12}(1-b_{21}\overline{U})}{1-\overline{U}}+1\right)\\ &= \overline{U}(1-\overline{U})\left(f(\overline{U})-\frac{b_{12}(1-b_{21}\overline{U})}{1-\overline{U}}\right),\end{aligned}$$

where $f(\overline{U})$ has the same previous formula with $\overline{U}\in(1/b_{31},1/b_{21})$ for this case. Indeed, the last term in the above expression is negative inside this domain of $\overline{U}$. Also,

$$\sup_{\overline{U}\in(1/b_{31},1/b_{21})} f(\overline{U}) = f(1/b_{31}).$$

Similar to that of Case 1, condition (14) implies that $L_u[\overline{U},\overline{V},\overline{W}](z)\le 0$ for this case.

Case 3: When $z>z_2$. By the previous case, we have $L_v[\overline{U},\overline{V}](z)\le 0$. Similarly, by condition (13), we can get $L_w[\overline{U},\overline{W}](z)\le 0$. For the U-equation, taking into account that $\overline{U}\in[0,1/b_{31})$, we have

$$\begin{aligned}&L_u[\overline{U},\overline{V},\overline{W}]\\ &\quad= \overline{U}''+c\overline{U}'+\overline{U}(1-b_{12}-b_{13}-\overline{U}+b_{12}b_{21}\overline{U}+b_{13}b_{31}\overline{U})\\ &\quad= \overline{U}(1-\overline{U})\left(\mu_1^2-\frac{\mu_1^2\overline{U}}{2(1-\overline{U})}-\frac{c\mu_1}{(1-\overline{U})^{\frac{1}{2}}}-\frac{b_{12}(1-b_{21}\overline{U})}{1-\overline{U}}-\frac{b_{13}(1-b_{31}\overline{U})}{1-\overline{U}}+1\right)\\ &\quad= \overline{U}(1-\overline{U})g(\overline{U}),\end{aligned}$$

where

$$g(\overline{U}) = 2-b_{12}-b_{13}-\frac{(1-b_{12}-b_{13})\overline{U}}{2(1-\overline{U})}-\frac{2(1-b_{12}-b_{13})}{(1-\overline{U})^{\frac{1}{2}}}-\frac{b_{12}(1-b_{21}\overline{U})+b_{13}(1-b_{31}\overline{U})}{1-\overline{U}}.$$

We compute

$$g'(\overline{U}) = \frac{1}{(1-\overline{U})^2}\left\{-(1-b_{12}-b_{13})\left(\frac{1}{2}+(1-\overline{U})^{\frac{1}{2}}\right)+b_{12}(b_{21}-1)+b_{13}(b_{31}-1)\right\}.$$

Since $(1-\overline{U})^{\frac{1}{2}}>(1-1/b_{31})^{\frac{1}{2}}$, condition (15) implies that $g'(\overline{U})\le 0$. Hence, maximum value of $g(\overline{U})$ is $g(0)=0$ and $L_u[\overline{U},\overline{V},\overline{W}](z)\le 0$, for all $z>z_2$.

Therefore, $(\overline{U},\overline{V},\overline{W})(z)$, with $c=c_0$, is a monotone upper solution to the system (5) where $\overline{U}(-\infty)=1$ and $\overline{U}(+\infty)=0$. By Lemma 1, we conclude that $c_{\min}$ is linearly determined. □

In view of the conditions (7)-(10), the inequalities of (8) and (10) have restrictions on the growth rates α_j, $j = 2,3$. Since our result in Theorem 1 is independent of α_j, we compare it with the related conditions (7) and (9). Actually, none of these conditions covers the other in general. To show the contribution of our result, we give a numeric example choices of the parameters. Let $b_{12} = 0.4$, $b_{13} = 0.3$, $b_{21} = 1.43$, and $b_{31} = 1.5$. By computations, conditions (7) and (9) are not satisfied while our new conditions (14)-(15) hold true.

Remark 1. *As a concrete choice, we may give an explicit formula for the upper solution $\overline{U}(z)$ in (16) as*

$$\overline{U}(z) = \begin{cases} 1, & z \le z_0, \\ 1 - \tanh^2\left(\frac{\mu_1}{2}(z_0 - z)\right), & z > z_0. \end{cases}$$

It is clear that the function is continuous and differentiable on $(-\infty, +\infty)$, and satisfies the problem (16). Here, z_0 is a shifting arbitrary constant.

Remark 2. *In the condition (15), taking $b_{\max}$ is an advantage so that it covers similar forms that are written in terms of b_{21} or b_{31}. In proof of Theorem 1, the condition is used with $b_{\max} = b_{31}$ as we consider the case $b_{21} \le b_{31}$.*

Theorem 2. *When (3) and (13) hold, the minimal speed of the system (5) is linearly determined if the following condition satisfies:*

$$\alpha_j \le \frac{(2 - d_j)(1 - b_{12} - b_{13})}{b_{j1} - 1}, \text{ for } j = 2,3. \tag{17}$$

Proof. Re-define $0 \le \overline{U}(z), \overline{V}(z), \overline{W}(z) \le 1$ as the following monotone, continuous, and differentiable function

$$\overline{U}(z) = \overline{V}(z) = \overline{W}(z) = \frac{1}{1 + Ae^{\mu_1 z}},$$

where μ_1 is defined in (12) and A is a constant. Similar to the proof of the above theorem, we prove that $(\overline{U}, \overline{V}, \overline{W})(z)$ is an upper solution for the system (5) when $c = c_0$. We substitute and use $\overline{U}' = -\mu_1 \overline{U}(1 - \overline{U})$, $\overline{U}'' = \mu_1^2 \overline{U}(1 - \overline{U})(1 - 2\overline{U})$, to get

$$\begin{aligned}
L_u[\overline{U}, \overline{V}, \overline{W}] &= \overline{U}'' + c\overline{U}' + \overline{U}(1 - \overline{U} - b_{12}(1 - \overline{U}) - b_{13}(1 - \overline{U})) \\
&= \overline{U}(1 - \overline{U})\left(\mu_1^2(1 - 2\overline{U}) - c\mu_1 + 1 - b_{12} - b_{13}\right) \\
&= -2\mu_1^2 \overline{U}^2 (1 - \overline{U}) \\
&\le 0, \\
L_v[\overline{U}, \overline{V}] &= d_2 \overline{V}'' + c\overline{V}' + \alpha_2 \overline{U}(1 - \overline{U})(b_{21} - 1) \\
&= \overline{U}(1 - \overline{U})\left(d_2 \mu_1^2 (1 - 2\overline{U}) - c\mu_1 + \alpha_2(b_{21} - 1)\right) \\
&= \overline{U}(1 - \overline{U})\left((d_2 - 2)(1 - b_{12} - b_{13}) - 2d_2 \mu_1^2 \overline{U} + \alpha_2(b_{21} - 1)\right) \\
&\le 0,
\end{aligned}$$

and $L_w[\overline{U}, \overline{W}] \le 0$ (similar to L_v). This completes the proof. □

Even though the second part of the condition (8) covers the condition (17), as $b_{j1}(b_{12} + b_{13}) < b_{j1}$, the contribution of the above theorem is notable as we don't require the restriction $b_{j1}(b_{12} + b_{13}) > 1$ of the first part of (8). Indeed, if the last inequality is not satisfied for both $j = 2$ and $j = 3$, the minimal speed is linearly determined by (7). But no results were given previously if $b_{j1}(b_{12} + b_{13}) > 1$ for only one of $j = 2$ or $j = 3$. For example, when $d_2 = d_3 = 1$, $\alpha_2 = 0.3$, $\alpha_3 = 0.7$, $b_{12} = 0.3$, $b_{13} = 0.4$, $b_{21} = 1.8$, and $b_{31} = 1.4$, we have $b_{21}(b_{12} + b_{13}) = 1.26 > 1$ and $b_{31}(b_{12} + b_{13}) = 0.98 < 1$, that is, (7) and (8) are not satisfied (as well as (9)), but our condition (17) implies the linear speed determinacy.

Comparing with the condition (10), we don't restrict the values of b_{12} and b_{13}. In fact, that restriction in the first part of (10) means

$$b_{1j} < 1 - b_{12} - b_{13}, \text{ for } j = 2, 3.$$

If this is not satisfied, then the condition (10) does not work, and our condition (17) covers the second part of it. This leads to combining both of them in a single condition as follows (we may use the above numeric choices as an example).

Corollary 1. *When (3) and (13) hold, the minimal speed is linearly determined if the following condition holds for $j = 2, 3$:*

$$\alpha_j \leq \frac{(2-d_j)(1-b_{12}-b_{13})}{M_j b_{j1} - 1}, \quad \textit{where } M_j = \min\{1, b_{1j}/(1-b_{12}-b_{13})\}.$$

Moreover, when (13) hold, the condition (11) implies that

$$\alpha_j \leq \frac{(2-d_j)(1-b_{12}-b_{13})^2}{\frac{1}{2}b_{j1}(b_{12}+b_{13}) - (1-b_{12}-b_{13})}, \text{ for } j = 2, 3,$$

which is covered by the condition in Corollary 1 when

$$M_j \leq \frac{b_{12}+b_{13}}{2(1-b_{12}-b_{13})}, \text{ for any } j = 2 \text{ or } j = 3.$$

Theorems 3-4 below give the linear speed determination for a larger domain of α_2, α_3, or both of them compared with that in Theorem 2.

Theorem 3. *When (3) and (13) hold, the minimal speed of the system (5) is linearly determined if the following conditions satisfy:*

$$3b_{13} + 2b_{12} \leq 2, \tag{18}$$

$$\alpha_2 \leq \frac{(2-d_2)(1-b_{12}-b_{13})}{b_{21}-1}, \tag{19}$$

$$\alpha_3 \leq \frac{2(2-d_3)(1-b_{12}-b_{13})}{b_{31}-1}. \tag{20}$$

Proof. Use the same definitions of $\overline{U}$ and $\overline{V}$ in the proof of Theorem 2,

$$\overline{U}(z) = \overline{V}(z) = \frac{1}{1+Ae^{\mu_1 z}},$$

and let

$$\overline{W}(z) = 1 - (1-\overline{U}(z))^2,$$

where μ_1 is defined in (12) and A is a constant. In addition to the derivatives of $\overline{U}$ given previously, we present the following derivatives of $\overline{W}$:

$$\overline{W}' = -2\mu_1\overline{U}(1-\overline{U})^2 \quad \text{and} \quad \overline{W}'' = 2\mu_1^2\overline{U}(1-\overline{U})^2(1-3\overline{U}).$$

The first derivative of the function $\overline{W}$ shows its monotonic. To prove that $(\overline{U}, \overline{V}, \overline{W})(z)$ is an upper solution for the system (5) when $c = c_0$, we substitute and use condition (18) to get

$$\begin{aligned} L_u[\overline{U}, \overline{V}, \overline{W}] &= \overline{U}(1-\overline{U})(\mu_1^2(1-2\overline{U}) - c\mu_1 + 1 - b_{12} - b_{13} + b_{13}\overline{U}) \\ &= \overline{U}^2(1-\overline{U})(b_{13} - 2(1-b_{12}-b_{13})) \\ &= \overline{U}^2(1-\overline{U})(3b_{13} + 2b_{12} - 2) \\ &\leq 0. \end{aligned}$$

Also, $L_v[\overline{U},\overline{V}] \leq 0$ is valid by the proof of the previous theorem. Finally, from conditions (13) and (20), we get

$$\begin{aligned} L_w[\overline{U},\overline{W}] &= \overline{U}(1-\overline{U})^2(2d_3\mu_1^2(1-3\overline{U}) - 2c\mu_1 + \alpha_3(b_{31} - 2 + \overline{U})) \\ &\leq \overline{U}(1-\overline{U})^2(2(1-b_{12}-b_{13})(d_3-2) + \alpha_3(b_{31}-1)) \\ &\leq 0. \end{aligned}$$

Lemma 1 completes the proof. □

Analogously to the above proof, by switching the definitions of $\overline{V}$ and $\overline{W}$ or by re-defining $\overline{V} = \overline{W} = 1-(1-\overline{U})^2$, the linear speed selection can be verified for another two sets of conditions. We present this in the following theorem.

Theorem 4. *When (3) and (13) hold, the minimal speed of the system (5) is linearly determined if one of the following sets of conditions satisfies:*

$$(1)\quad \begin{cases} 2b_{13} + 3b_{12} \leq 2, \\ \alpha_2 \leq \dfrac{2(2-d_2)(1-b_{12}-b_{13})}{b_{21}-1}, \\ \alpha_3 \leq \dfrac{(2-d_3)(1-b_{12}-b_{13})}{b_{31}-1}. \end{cases} \qquad (2)\quad \begin{cases} 3b_{13} + 3b_{12} \leq 2, \\ \alpha_2 \leq \dfrac{2(2-d_2)(1-b_{12}-b_{13})}{b_{21}-1}, \\ \alpha_3 \leq \dfrac{2(2-d_3)(1-b_{12}-b_{13})}{b_{31}-1}. \end{cases}$$

4 Conclusions

We considered the biological invasion of two resident species by a strong competitor via the three-species competition model (1). The minimal speed determinacy of the traveling wave solution was investigated. We have first transformed the model into the cooperative non-dimensional model (2). Then, by applying the upper-lower solution method on (2) when (3) holds, we derived new sufficient conditions so that the minimal speed of the traveling wave solution is linearly determined. See Theorems 1-4.

A condition that is independent of the growth rates, i.e., for any positive α_2 and α_3, was provided and compared with the previous related conditions (7) and (9). We also derived new conditions based on the problem parameters, including the growth rates. By comparing these conditions with that in previous works, we showed the contribution of our results and some improvements were discussed. Some previous conditions on the linear speed determination were combined with ours to derive a more general result. See Corollary 1.

The invader species in the model and the associated assumptions is assumed to be a strong competitor and shall win the competition. Our results determine its spreading speed toward the winning state based on the species' growth, dispersal, and competition strength. For some interactions, these factors can be estimated experimentally. Then the species spreading speed can be found when one of the given conditions holds. Also, the formulas given in this study play a significant role in the controlled biological interactions, e.g., agricultural pest control. Based on the controlled parameters, one can expect the rate of spreading.

Actually, the upper-lower solution method plays a main role in deriving significant results for such studies. This methodology can also be used to study multiple-species models. As there is no sufficient and necessary condition for the minimal speed determinacy, new upper or lower solutions may be constructed, taking care about their properties, or other methods may be used to contribute on the problem.

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